Nonlinear R-T.

- Layzer Solution
- Bubble Parametrization
- Multi-Bubbles / Mix

Lect. 5 - 6 - 7

I.) Nonlinear Rayleigh-Taylor Instability: Single Mode / Bubble

Ref. D. Layzer Ap. J. 122 1 (1955). (a must)
H. J. Kull, Review

A.) Motivation and Heuristics

Recall;

i.) in linear phase, simple R.T. instability

$\gamma = \sqrt{gAk}$ $(k\eta \ll 1)$

ii.) in nonlinear phase expect algebraic growth
(i.e. simple intuition) $\rightarrow$  bubble rise $z(t)$

$\ell = \alpha A g t^2$ $(k\eta \gtrsim 1)$

Seek: $\rightarrow$ how recover algebraic growth? {light rises a la bubble}

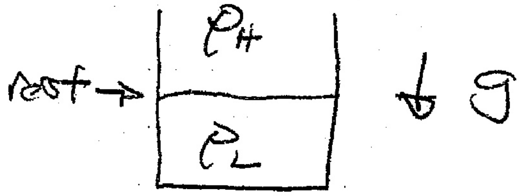
$\rightarrow$ how unify linear, nonlinear regimes?

$\rightarrow$ understand flow structure in nonlinear regime

Heuristically;

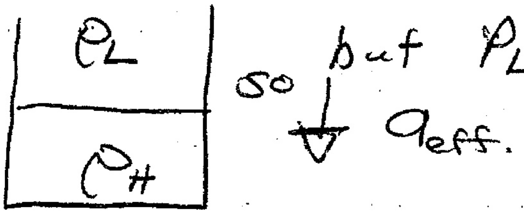
→ HW on Tuesday
→ no note borrowing on Tues, Th. A.M.

Note equivalence:



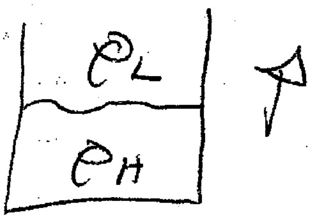
to:

g ignored.



so but $\rho_L \gg \rho_H$

in frame of interface/membrane, equivalent to



$g_{eff} = -a_{eff.}$
(same as pedagogical example)

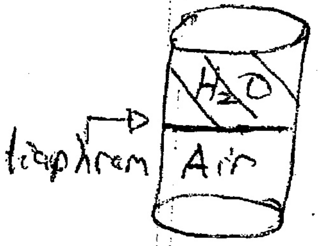
In general:

"R-T" occurs when light fluid 'accelerated into/toward' heavy fluid.

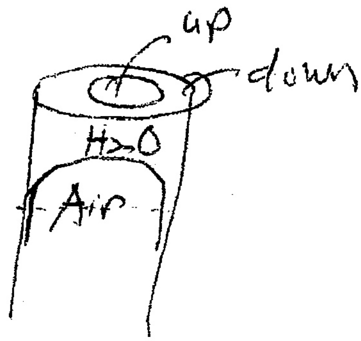
"NL": $\nabla\phi \cdot \nabla\eta \sim \frac{\partial\phi}{\partial z}$

$\Rightarrow k\eta \sim 1$.

then, for cylinder:



$\Rightarrow$



$A = 1$

i.e. $\begin{cases} \text{air bubble rises in cente} \\ H_2O \text{ spike falls at edge} \end{cases}$

Net mass flux thru dia $= 0$ $\Rightarrow$ "spike and bubble" picture R, i

$\therefore$ η (interface variable) $\rightarrow$ R (bubble radius)

(field, fourier modes) (structure)

then, linear theory $\Rightarrow$ $R \rightarrow \ell$

$$\gamma = \frac{1}{R}\frac{dR}{dt} = \sqrt{kg}$$

$$\frac{dR}{dt} = V = \sqrt{(kR)gR}$$

bubble rize velocity

∴ natural to delineate:

① $k_1 R = k\eta < 1$ — linear
$\Rightarrow R = R(0)\, e^{\gamma t}$

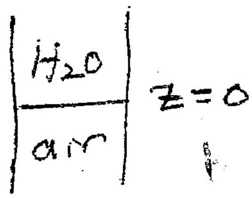
② $k_1 R \gtrsim 1$ $\quad V = \sqrt{gR}$ — nonlinear

i.e. $\quad \dfrac{dR}{dt} = \sqrt{gR} \Rightarrow \underline{R = \alpha g t^2}$

Suggests can understand nonlinear R.T. via notions of bubble dynamics!

B.) D. Layzer Calculation

Consider tube:



Units: $\quad \underset{\substack{\text{tube radius}\\ \downarrow}}{R} / \beta_1 = 1$ $\qquad J_1(r)\Big|_{r=\beta_1} = 0$
(first zero)

$g = 1$

$A = 1$

B.C's: Hard wall: $\quad V_r(z, \beta_1, t) = 0$

$\Rightarrow \partial_r \phi(z, \beta_1, t) = 0$

Evanescent at top: $V_z(z,r,t)\Big|_{z\to\infty} = 0$

$$\Rightarrow \quad \partial_z \phi(z,r,t)\Big|_{z\to\infty} = 0$$

Potential Flow: $\nabla^2\phi = 0$

Approach to solution: → How get/see ('spike and bubble') picture? shape – streamlines

→ convert interface dynamics problem into (nonlinear) particle mechanics problem

→ use fluid particle eqns. of motion to determine stream-lines, equation for boundary

Now, general solution to $\begin{cases} \text{Laplace eqn.} \\ \text{B.C.'s} \end{cases}$

$$\phi = \underbrace{F(t)}_{\text{arbitrary fctn time}} e^{-z} J_0(r)$$

→ does not satisfy Bernoulli Eqn.!

Then, for fluid particle:

$$\frac{dx}{\partial_x\phi} = \frac{dy}{\partial_y\phi} = \frac{\partial z}{\partial_z\phi}$$ → equations for streamline

$$\frac{dr}{dt} = V_r = -\partial_r \phi$$
$$= + F(t) e^{-z} J_1(r)$$

$$\frac{dz}{dt} = V_z = -\partial_z \phi$$
$$= + F(t) e^{-z} J_0(r)$$

(Layzer notation)

exploits potential flow structure of problem $\rightarrow$ obviously not universally applicable

Then, for stream-lines, can write:

$$dz/dr = V_z / V_r$$
$$= (dz/dt)/(dr/dt)$$
$$= J_0(r)/J_1(r)$$

but Bessel identity $\Rightarrow$ $J_1'(r) = J_0(r) - \frac{J_1(r)}{r}$

$\therefore$

$$\frac{dz}{dr} = \frac{J_1'(r) + J_1(r)/r}{J_1(r)}$$

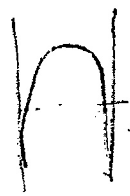
$$= \frac{J_1'}{J_1} + \frac{1}{r}$$

$$dz = \frac{J_1'}{J_1}\,dr + \frac{dr}{r}$$

$$z = \ln(J_1(r)) + \ln r$$

$$\Rightarrow \boxed{e^{z} = C\, r\, J_1(r)} \rightarrow$$ parametrizes fluid streamlines, if interface ② slightly distorted. (no Bernoulli, yet)

Note:

- generates stream surfaces via displacement along $\hat{z}$
- stream surfaces like:  → asymptotic to z axis

$$\Rightarrow \text{i.e. } \left. e^{-|z|}\right|_{z\to-\infty} \sim \left. c\, r\, J_1(r)\right|_{r\to R}$$

⇒ streamline structure is under-pinning of spike + bubble intuition!

⇒ not really solutions, but effect due to B.C. artifact (see 24a.)

Now, to obtain i.e. generic equation of motion for interface:

i.e.,

→ Solution not yet satisfy Bernoulli eqn ⟺ interface distortion

but

→ $\boxed{\text{message is that B.C.'s force spike structure}}$

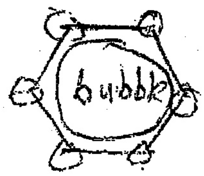
i.e. $1 = \underbrace{C_n J_1(r)}_{\text{small}}\, \underbrace{e^{-z}}_{\text{large}}$

i.e.

→ spike + bubble picture consequence $\left\{\text{B.C.'s symmetry}\right.$

i.e. hexagonal symmetry (ala' convection)

⇒



→ spikes at vertices
bubble at center

(top view):

⇒ integrate fluid equations of motion:

$$\frac{dr}{dt} = + F(t)\, e^{-z} J_1(r) \qquad \text{i.e.} \quad \frac{dx}{dt} = V$$

$$= \nabla\phi$$

$$\frac{dz}{dt} = + F(t)\, e^{-z} J_0(r)$$

define: $Z = e^{z}$ → vertical variable

$V = r^2$ → radial

$T(t) = \int_0^t dt\, F(t) + 1$ → time variable

$k(V) = 2J_1(r)/r$ → shape

Now,

$$\frac{dZ}{dt} = \frac{dz}{dt} e^{z} = + F(t)\, J_0(r) \qquad \text{(fluid eqns.)}$$

$$= + F(t)\, J_0(\sqrt{V})$$

⇒

$$\boxed{\frac{dZ}{dT} = J_0(\sqrt{V})}$$

and

$$\boxed{Z = V k(V) / (dV/dT)}$$

→ plus.

Eliminating Z :

$$\frac{V}{V_0} = (T-1)\,k(V_0)\,/\,Z_0+1$$

$$\frac{Z}{Z_0} = \frac{k(V)}{k(V_0)}\left(\frac{(T-1)\,k(V_0)}{Z_0+1}\right)$$

→ velocity

Now:

→ potential can't solve Bernoulli eqn. over full surface

∴ → seek expansion valid near bubble vertex, i.e. week distortion, see (26b)

→ un-perturbed surface flat $\left\{ \begin{array}{l} V_0 = 0 \\ Z_0 = 1 \end{array} \right.$ $r \simeq 0$ cylindric

{ flat surface approximation

∴ neglecting non-linearities in V $(r \simeq 0)$ ↳ expansion

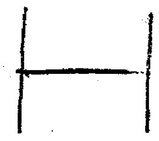
$$\Rightarrow \quad \frac{V}{V_0} = (T-1)\,\frac{k(V_0)}{Z_0+1}$$

$$= (T-1)\,\frac{2\,\cancel{r}}{2}$$

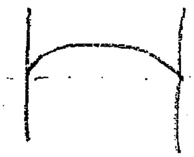
$$\therefore \quad V = V_0\,(T-1)$$

i.e.

→ unperturbed



→ weakly perturbed



→ strongly perturbed

$\Omega \simeq 0$ approx.

domain validity of treatment shrinks as bubble rises.

$\eta' \sim Z$

Similarly: $e^{\eta'} = (T-1)\left[1 - \frac{V}{8}(1-T^{-2})\right]$

$Z \sim \ln(T)$, $T \gg 1$

$(\eta' \sim Z)$

$\therefore \Rightarrow$ $\boxed{\begin{aligned} &V = V_0 (T-1) \\ &e^{\eta'} = (T-1)\left[1-\frac{V}{8}(1-T^{-2})\right] \end{aligned}}$

Then: $\phi = F(t)\, e^{-Z} J_0(r)$ into Bernoulli

$\frac{\partial \phi}{\partial t} - (\nabla\phi)^2 \eta = 0$

$\Rightarrow$ $\boxed{T(T^2+1)T'' - T'^2 - T^2(T^2-1) = 0}$

(coeff, $V=0$)

i.e. $\Rightarrow F \rightarrow T'$

$\Rightarrow \eta_1 = \eta_1(T, V)$ (above)

$\Rightarrow {}' = \frac{d}{dt}$ (Bernoulli in real time)

Check:

a.) Linear Regime (small time)

$T = 1 + \tau$, $\tau \ll 1$

$\Rightarrow (1+\tau)((1+\tau)^2+1)\tau'' - \tau'^2 - (1+\tau)^2((1+\tau)^2-1) = ($

$$T'' - 2T = 0$$

i.e. $T = T(0)\, e^{t}$ $\Rightarrow$ exponential growth a la' linear theory.

In dimensional units:

$$T = T(0)\, e^{\gamma t} \qquad \gamma = \sqrt{\frac{Rg}{\beta_1}}$$

$$\sim \sqrt{4g_{eff}}$$

$\Rightarrow$ set by cylindrical geometry

b.) Nonlinear Regime $\quad T \gg 1$

$$T^3 T'' - \cancel{T'^2} - T^4 = 0$$

$$T'' - T = 0$$

$\Rightarrow \quad T = e^{t} \quad \Rightarrow \quad T' = F(t) = e^{t}$

$$\underset{\text{Bubble rise}}{V} = \left.\frac{d\eta}{dt}\right|_{r=0} = \frac{d}{dt} \frac{\ln(T)}{T'}$$

$$= \frac{1}{T} \frac{dT''}{dt} = 1$$

(interface eqn.)

$e^{\eta} = (T-1)$ (mod)

$\dot{\eta} e^{\eta} = \dot{T} \quad \eta \sim \ln T$

$T \gg 1$

Physics 218A P. Diamond

Bubble, Bubble, Toil and Trouble, ...

Last time:

→ discussed D. Layzer('55) solution for single, nonlinear bubble.
- why: observables in NL state
$\gamma^{-1}(1cm) \sim .1$ sec.

- what: → single mode ($\lambda \leftrightarrow R$) growth in $\begin{cases} k\eta \sim 1 & (\eta/R \sim 1) \\ t \to \infty \end{cases}$ regime
i.e. $V = \#(gR)^{1/2}$
→ connects exponential and algebraic growth regimes (NL saturation).

- how: → flat interface approximation at (i.e. $\Gamma \to 0$) ⟷ geometry bubble tip.
→ use of streamlines from $\nabla^2\phi = 0$ and (assumed) self-similarity (valid at bubble tip).

→ Today:
(i.) Lazy Man's Layzer
(ii.) Bubble Competition

I.) Recall basic R-T. equations:

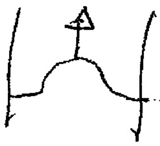
- $\nabla^2 \phi = 0$

- $\frac{\partial \eta}{\partial t} + \underline{\nabla}\phi \cdot \underline{\nabla}\eta = V_z$ or, more generally, "interface moves with fluid"

- $\frac{\partial \phi}{\partial t} + \underline{\nabla}\phi \cdot \underline{\nabla}\phi + g\eta = \text{const.}$

For single mode (bubble):

$$\phi(x, z, t) = a(t)\cos(kx)\, e^{-kz}$$


→ since bubble is rising light fluid, bubble flow must vanish at $+\infty$.

Now, essence of Layzer / bubble tip approximation is geometric ↔ assumed shape of bubble, i.e.:

$$z_i(x,t) = \eta(x,t) = z_0 + z_1 (x - x_i)^2$$

i.e. parabolic shape approximation:

$x_i, z_0 \rightarrow$ tip location

$z_1 \rightarrow$ radius of curvature of bubble

i.e. $R_c = -1/2z_1$

Note: Linear: $\gamma \tilde{\eta} = \frac{\partial \tilde{\phi}}{\partial z}$

i.e. $\tilde{\phi}$ forces η

Layzer: $\eta \leftrightarrow$ geometry

Then:

$$\eta = z = z_0 + z_1 (x - x_1)^2$$

so

$$\frac{\partial \eta}{\partial t} + \underline{V} \cdot \underline{\nabla} \eta = \frac{\partial \phi}{\partial z}$$

$$\Rightarrow \frac{dz_0}{dt} + (x - x_1)^2 \frac{dz_1}{dt} + (-2) z_1 (x - x_1) \frac{dx_1}{dt}$$

$$+ V_x \, 2 (x - x_1) = V_z$$

$$\Rightarrow \boxed{V_z - \frac{dz_0}{dt} - \frac{dz_1}{dt}(x-x_1)^2 - 2z_1(x-x_1)\left(V_x - \frac{dx_1}{dt}\right) = 0}$$

$\therefore$ have:

$$\phi = a(t)\cos(kx)\,e^{-kz} \quad ; \quad \underline{V} = \nabla\phi$$

(a)
$$V_z - \frac{dz_0}{dt} - \frac{dz_1}{dt}(x-x_1)^2 = -2z_1(x-x_1)\left(V_x - \frac{dx_1}{dt}\right)$$
interface eqn.

and

(b)
$$\frac{\partial\phi}{\partial t} + \frac{V_x^2 + V_z^2}{2} + gz = \text{const.}$$

Plugging in ϕ to 2nd order:

Interface eqn: $O(a)$; for x

$$-ka(t)\underbrace{(\cos kx)\Big|_{x=0\,(\text{tip})}}_{1} e^{-kz_0} - \frac{dz_0}{dt} = 0$$

$$\Rightarrow \boxed{a k e^{-k z_0} + \frac{d z_0}{dt} = 0}$$

to O(2):

$$\boxed{a k^2 \left(z_1 + \frac{k}{2} \right) e^{-k z_0} - \frac{d z_1}{dt} + 2 z_1 a k^2 e^{-k z_0} = 0}$$

and Bernoulli (to O(2)):

$$\boxed{\underset{\downarrow\ \partial\phi/\partial t}{k e^{-k z_0} \left(z_1 + \frac{k}{2} \right) \frac{da}{dt}} + \underset{v^2}{a^2 k^3} z_1 e^{-2 k z_0} - g z_1 = 0}$$

Recall : $\rightarrow e^{-k z}$, $z \rightarrow z_0 + z_1 (x - x_1^2)$ HW.

$\rightarrow$ bubble tip $(x \rightarrow 0)$

$\Rightarrow$ equiv. Layzer Solution.

Check : Linear solution:

$$\frac{k^2}{2} e^{-k z_0} \frac{da}{dt} = g z_1 \qquad (1)$$

②

$$\frac{a k^3}{2} e^{-kz_0} = \frac{dz_1}{dt}$$

so, small perturbations $\Rightarrow$ d/dt ①

$$\frac{k^2}{2} e^{-kz_0} \frac{d^2 a}{dt^2} = \frac{g a k^3}{2}$$

$$\frac{d^2 a}{dt^2} = g k a \quad \checkmark, \text{ etc.}$$

Now, for late times: Remark

$$k e^{-kz_0}\left(z_1 + \frac{k}{2}\right) \cancel{\frac{da}{dt}} + a^2 k^3 z_1 e^{-2kz_0} = g z_1$$

mode sats. $\left\{ v^2 \sim g\eta \sim g\lambda \right.$ balance

$$\Rightarrow a^2 k^3 \sim g$$

$$(ka)^2 \sim g/k \quad \Rightarrow \quad \boxed{v^2 \sim g\lambda}$$

akin $v^2 \sim (gR)$

etc.

Then $V = 1$, or in dimensional units:

$$\boxed{V = \left(\frac{gR}{\beta_1}\right)^{1/2}}$$

Agrees with free-fall intuition

c.) Can find general vertex dynamics solution.

d.) $\left.\begin{array}{l} T \sim 1 \\ T \gg 1 \end{array}\right\}$ limits establish $k\eta \sim 1$

as criterion for entrance into nonlinear regime.

e.) Heuristics for Multiple Mode Systems

- Layzer solution for single bubble/mode
- in reality, ICF target finish irregularities initialize **many** modes

Seek: multi-mode criterion for non-linearity
⇒ amplitude for exponential growth cessation ?!

ref: S. Haan, Phys. Rev. A **39** 5812 (1989).

First, observe: all modes can't grow to $k\eta_{\underline{k}} \sim 1$
$\Rightarrow$ spectral content diverges!,

i.e.

$$\langle \xi^2 \rangle = \int d\underline{k}\, |\eta_{\underline{k}}|^2$$

$$= \int k\, dk\, \frac{1}{k^2}$$

$$\to \infty$$

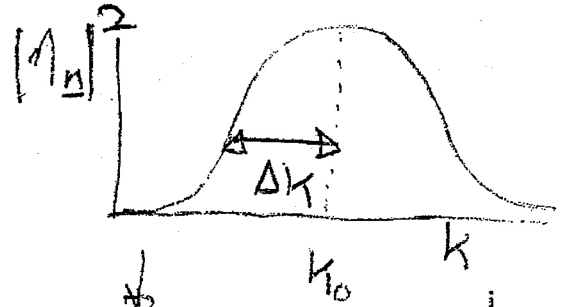
$\boxed{\text{Some modes, especially long wavelength, remain in linear regime (slow)}}$

$\therefore$ natural, in multi-mode case, to suggest nonlinearity criterion $k_0\eta_{rms} \sim 1$, where

η_{rms} from superposition modes

k_0 refers to wave number of spectral peak

i.e.



{ spectrum $\to$ peak + width

long wavelength modes grow slowly

ablation $\hookrightarrow$ cut-off

$$\therefore \quad k_0^{-2} = \frac{L^2}{(2\pi)^2}\int_0^\infty 2\pi k\, dk\, |\tilde{\eta}_k|^2 = \langle \tilde{\eta}^2 \rangle_{MT}$$

$$= \frac{L^3}{2\pi} k_0 \Delta k\, |\tilde{\eta}_{k_0}|^2$$

$\Rightarrow$ establishes criterion:

$$\eta_{k_0} \cong \frac{\sqrt{2\pi}}{L}\left(k_0^3 \Delta k\right)^{-1/2}$$

if $\Delta k \sim k_0$ (frequent state affairs)

$\Rightarrow$

$$\boxed{\eta_{k_0} \cong \frac{\sqrt{2\pi}}{L k_0^{+2}}}$$

contrast to $\eta_{k_0} \sim 1/k_0$

$\Rightarrow$ scaling with system!

$\rightarrow$ in multimode system, superposition (bubble competition) of many-mode interface displacements $\Rightarrow k_0 \eta_{k_0} \cong \frac{\lambda_0}{L}$, not 1 (a la single wave)

$\rightarrow$ transition to Layzer regime at lower amplitude.

$\rightarrow$ consistent with LLNL simulations, experiments

e) Bubble Competition and Mix $\leftrightarrow$ Heuristics

$\rightarrow$ goal is to:

- construct model of nonlinear Rayleigh-Taylor using Layzer spike + bubble model

- generate model of growth of mixing layer

$\rightarrow$ ingredients:

- single bubble model

- bubble - bubble interaction $\Rightarrow$ Competition

A.) Single Bubble

- Layzer calculation $\Rightarrow$ bubble vertex rises at $V = (gR/\beta_1)^{1/2}$

$\therefore$ suggests that mixing layer grows as:

$$\frac{d\ell}{dt} = (g\ell)^{1/2} \qquad , \quad \ell \sim \alpha g t^2$$

but

omits other effects which limit bubble / spike dynamics:

→ compression

→ drag, spike KH

1 D Model: $\begin{cases} V(\ell, z, t) \rightarrow \text{bubble rise velocity} \\ \ell(z, t) \rightarrow \text{mixing layer width} \end{cases}$ (z

$$\frac{\partial \ell}{\partial t} + \underbrace{V \frac{\partial \ell}{\partial z}}_{\text{bubble compression}} = \underbrace{V}_{\text{bubble rise velocity} \sim (g\ell)^{1/2}}$$

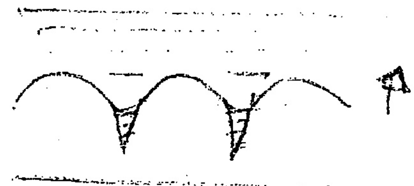
$$\rho \frac{\partial V}{\partial t} = \underbrace{\rho g}_{\text{gravitational force}} - \rho C_D \frac{V^2}{\ell} \rightarrow \underline{\text{drag}}$$

D. Youngs
Physica D 37 (198)
270
+ refs therein

Physics of Drag:

- view bubble rise as mixture of two inter-penetrating fluids:

- i.e.



{inter-penetrating fluids

→ drag coefficient (phenomenological)

$$F_D = C_D \frac{V}{\ell} \underbrace{\rho V}_{\text{momentum}}$$

↓ time scale ($\ell \leftrightarrow$ "chunk" size for interpenetrating fluids)

Drag slows bubble rise

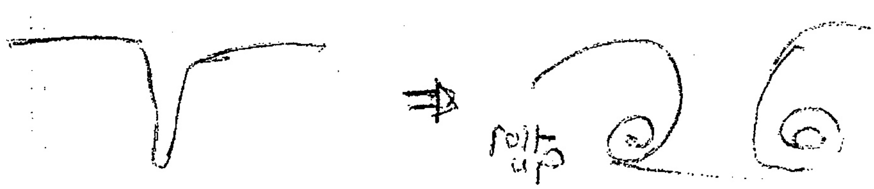
i.e. $\frac{\partial V}{\partial t} = 0 \Rightarrow V = \left(\frac{1}{C_D}\right)^{1/2} (g\ell)^{1/2}$

$$\boxed{\ell = C_D^{-1/2} \frac{g}{2} t^2}$$

∴ Drag coefficient contains physics of $\alpha \approx .05$ scaling!

- alternatively, view drag term as manifestation of decay due to spike shear flow instability,
  i.e.: recall:

2D only. 2D



roll up

→ blunt tip, slows spike fall

8A ?,b

Now, time scale for roll-up is flow shear rate,

i.e. $\gamma \sim |\nabla u|$

$\therefore$ dimensionally, $\gamma \sim \frac{V}{\ell}$ $\Rightarrow$ rate of drag.

$\Rightarrow$ Drag $\left\{ \begin{array}{c} \text{term} \\ \text{coefficient} \end{array} \right\}$ essential to proper fit of implosion experiment data!

B.) Bubble Competition

observe:

- single bubble mixing model characterized by single length scale

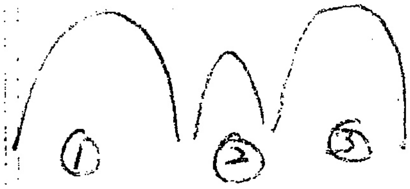
but

- multi-mode system $\Rightarrow$ multi-bubbles

$\therefore$

- need describe trend in evolution of bubble length scale

Simple example:



$\underline{t}$ 3 bubbles
exterior larger

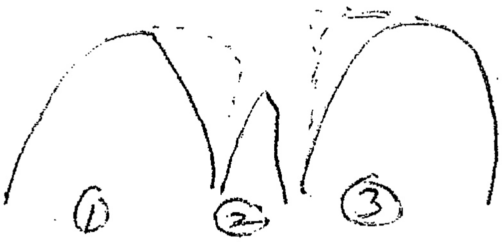
$\ell_{2} < \ell_{1}, \ell_{3}$

→ but, in Layzer regime:

$$v \sim (g\ell)^{1/2}$$

⇒ larger bubbles rise faster

→ ∴, at $t+\Delta t$



⇒ 1, 3 will tend to expand over 2, thus "squeezing" it out

→ trend in bubble competition is:

- larger bubbles rise faster
- squeeze out smaller bubbles

⇒ trend is progression towards single length scale characteristic of largest bubble
⇒ seen as "inverse cascade".

→ can build multi-bubble Layzer model (2D with x discrete, dynamics in z).

⇒ colloidal aggregation

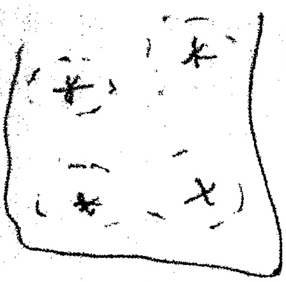
Why non diffusive?

→ process is self-accelerating ⇒ large eddys move faster, so more energy.

$\tau_{scp} \sim \ell^{2/3}/\epsilon^{1/3}$.

(2) Aggregation → "Inverse Cascade" Assembly.

Consider "sticky particles" (i.e. colloid) in (i.e. electrolytic) solution.



[i.e. each particle has a sphere of influence R (i.e. Debye length)]

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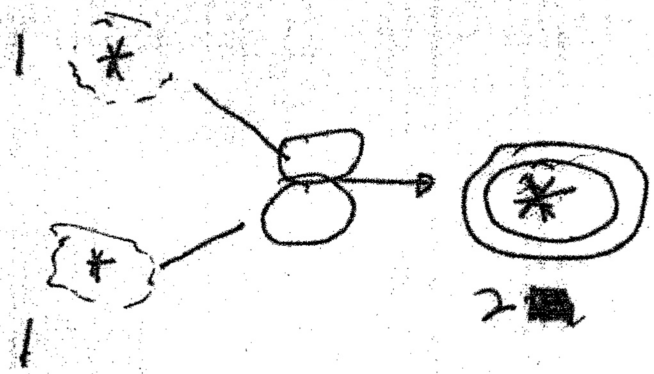
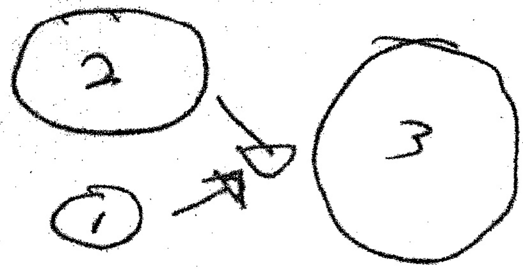
Now – particles move only by
· diffusive random walk
(i.e. Brownian motion)

– if two particles hit within R, they stick → form high order cluster.





What will happen?

– evolution
– fate, ultimate
– populations along way
– time scale.

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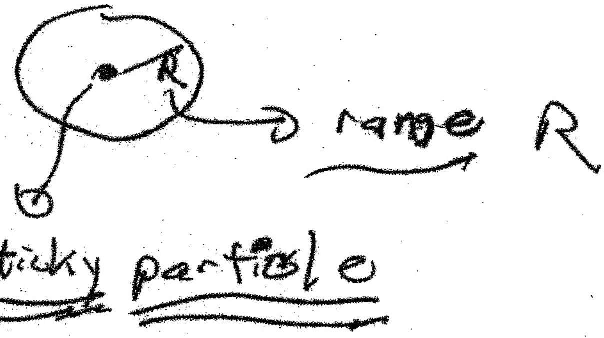
consider basic interaction

Now,



Consider sticking events

↓

random walk into sphere R

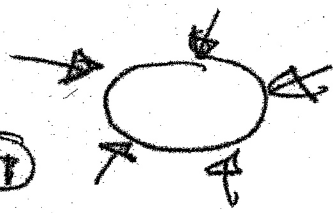
↓

① Sticking events ~ rate of arrivals at $R \sim R$ surface.

$$\frac{\partial n}{\partial t} + \underbrace{\nabla \cdot \Gamma}_{\text{diffusive flux}} = 0$$

continuity eqn. for particle

$$\Gamma \sim \underbrace{-D\nabla n}_{\text{(Fick)}} \sim \frac{Dn}{R}$$



so

$$\underbrace{\frac{\partial N}{\partial t}}_{\text{number/time}} \sim \overbrace{(4\pi R^2)}^{A}\,\overbrace{\frac{Dn}{R}}^{\Gamma}$$

$$\sim DRn$$

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$$\boxed{\left.\partial N/\partial t\right|_{\text{arrivals}} \sim D R n}$$

basic rate

(2) Now, all particles also undergo Brownian motion, too. So

$$\left.\partial N/\partial t\right|_{\text{arrivals}} \sim (\underbrace{D_1}_{\text{to diff}} + \underbrace{D_2}_{\text{of diff.}}) R n$$

Problem suggests a process of birth and death, which is hierarchical:

$\boxed{\text{birth}}$: i and j aggregate stick to form k
$\rightarrow$ k born

$\boxed{\text{death}}$: k and i aggregate to form $k+i$

$\rightarrow$ k dies

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N.B. Cascade is birth/death process.

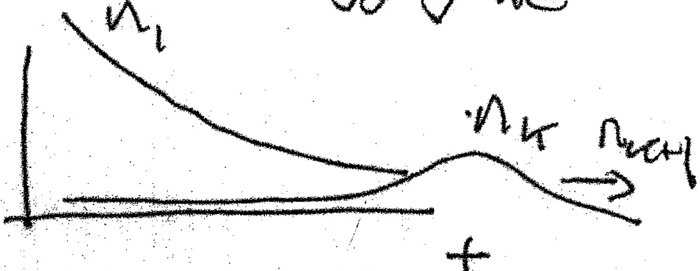
Can describe population evolution by

n_1

n_2 → populations of k^{th} aggregate

⋮

expect:



n_k

⋮

where, each n_k satisfies! i.e. higher aggregates grow sequentially. ⇒ and big one is ultimate state

$$\frac{dn_k}{dt} \sim \begin{pmatrix} \text{birth of} \\ k\text{-fold} \end{pmatrix} - \begin{pmatrix} \text{death of} \\ k\text{-fold} \end{pmatrix}$$

birth: all $i+j$ s/t $i+j=k$

$$\left.\frac{dn_k}{dt}\right|_{\text{birth}} \sim \sum_{\substack{i,j \\ i+j=k}} n_j \,(\text{rate arrival } i)$$

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$$\frac{dn_k}{dt} \sim \sum_{\substack{i,j \\ =k}} n_j \, (D_{i,j} R_{i,j}) \, n_i$$

symmetrize → i,j arbitrary

$$\sim \sum_{\substack{i,j \\ i+j=k}} \frac{n_i}{2} n_j \, D_{ij} \, R_{ij}$$

$D_{i,j} = D_i + D_j$ → rndm motion independent

R_{ij} = radius influence of aggregation $\sim R_i + R_j$

Likewise:

$$\left.\frac{dn_k}{dt}\right)_{\text{death}} \sim k + \text{any to form higher aggregates.}$$

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$$\left.\frac{dn_k}{dt}\right)_{\text{death}} \sim -n_k \overline{\sum_i} D_{k,i} R_{k,i} n_i$$

so, for all n_k:

$$\boxed{\frac{dn_k}{dt} \sim \underbrace{\frac{1}{2}\overline{\sum_{\substack{i,j \\ i+j=k}}} D_{ij} R_{ij} n_i n_j}_{\text{birth}} \underbrace{- \overline{\sum_i} D_{k,i} R_{k,i} n_i n_k}_{\text{death}}}$$

has generic form:

Master Eqn.

$$\frac{dn_k}{dt} = \underbrace{\overline{T_{in}}}_{T(i,j|k)} n_j - \underbrace{T_{out}}_{T(k,i|k+i)} n_k$$

Transition rates

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obviously

$$\tau \sim D_{n_i} R_{k_i} n_i$$

⇒ sets time scale.

Now recall for Brownian Motion:

$$D \sim k_B T / \underbrace{6\pi\eta a}_{\text{Stokes}}$$

i.e. $\tilde{v}^2 \tau$

$$\sim \frac{T}{m_p} \frac{m_p}{6\pi\eta a}.$$

Now note: $D \sim 1/R$

↳ scale of particle/structure

so at B-O-E level:

$RD \sim$ const. ⇒ $RD\, n_0$ sets time scale.

$$m\frac{dv}{dt} = -\beta v + \tilde{f}$$

$$\beta = 6\pi\eta a$$

$\tilde{v} \sim \tilde{f}/\beta$ at terminal velocity

$D \sim \langle \tilde{v}^2 \rangle \tau$

↳ memory time of v field.

$\sim \langle (\tilde{v})^2 \rangle \frac{m}{\beta}$

$\langle \tilde{v}^2 \rangle \sim T/m$, by thermal eqbrm.

$$D \sim k_b T / 6\pi\eta a$$

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so, normalize n to n_0, and

$$\tau = \underbrace{D R n_0}_{\text{Loss rate}} t$$

$\Rightarrow$

$$\boxed{\frac{dn_k}{dt} = \sum_{\substack{i,j \\ =k}} n_i n_j - 2 n_k \sum_i n_i}$$

To solve, note:

$$\sum_k n_k$$

$$\frac{d}{dt} \sum_k n_k = \sum_{ij} n_i n_j - 2 \sum_{i,k} n_k n_i$$

$$= -\left(\sum_{k=1} n_k\right)^2 \quad \text{re-label}$$

$$dx/dt = -x^2$$

$$\sum_k n_k \equiv n_0 \quad \text{i.c.}$$

$$\Rightarrow \sum_k n_k = n_0 \Big/ 1 + n_0 \tau$$

Now, for n_1

$$\frac{dn_1}{dt} = -2 n_1 \sum_{k=1}^{\infty} n_k$$

no n_0

$$= \frac{-2 n_1 n_0}{(1+n_0\tau)}$$

$\Rightarrow$

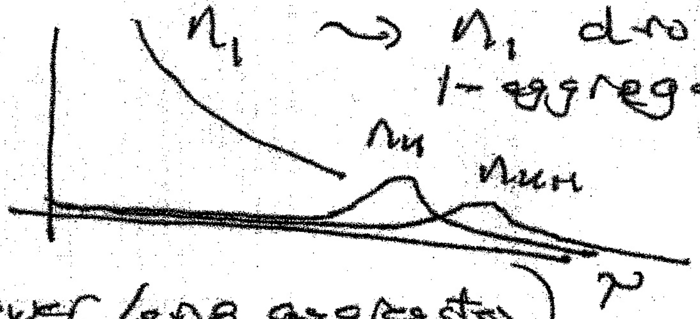
$n_1 = n_0 / (1+n_0\tau)^2$ and work up.

$$\boxed{n_k \sim n_0 \left[(n_0\tau)^{k-1} / (1+n_0\tau)^{k+1} \right]}$$

$\Rightarrow$

- particles aggregate to higher, higher k
- form: for higher k
- peak later
- lower (fewer large aggregates)

$n_1 \rightsquigarrow n_1$ drops as 1-aggregates absorbed.



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→ $n_{k\,max}$ eventually dominates.

→ Classic inverse cascade phenomenon

⇒ large structures sweep up small structures.

⇒ assemble into 1 large blob.

→ if separate: gelation

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